

All fractional (g, f) -factors in graphs ^{*}

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Abstract

Let G be a graph, and $g, f : V(G) \rightarrow N$ be two functions with $g(x) \leq f(x)$ for each vertex x in G . We say that G has all fractional (g, f) -factors if G includes a fractional r -factor for every $r : V(G) \rightarrow N$ such that $g(x) \leq r(x) \leq f(x)$ for each vertex x in G . Let H be a subgraph of G . We say that G admits all fractional (g, f) -factors including H if for every $r : V(G) \rightarrow N$ with $g(x) \leq r(x) \leq f(x)$ for each vertex x in G , G includes a fractional r -factor F_h with $h(e) = 1$ for any $e \in E(H)$, then we say that G admits all fractional (g, f) -factors including H , where $h : E(G) \rightarrow [0, 1]$ is the indicator function of F_h . In this paper, we obtain a characterization for the existence of all fractional (g, f) -factors including H and pose a sufficient condition for a graph to have all fractional (g, f) -factors including H .

Keywords: graph; fractional (g, f) -factor; all fractional (g, f) -factors.

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1 Introduction

We consider finite undirected graphs which have neither multiple edges nor loops. Let G be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For each $x \in V(G)$, the degree of x in G is defined as the number of edges which are adjacent to x and denoted by $d_G(x)$. For any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S , and use $G - S$ to denote the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S . A subset S of $V(G)$ is said to be independent if $N_G(S) \cap S = \emptyset$. Let S and T be two disjoint vertex subsets of G . Then $e_G(S, T)$ denotes the number of edges joining S to T .

Let $g, f : V(G) \rightarrow N$ be two functions with $g(x) \leq f(x)$ for each $x \in V(G)$. A spanning subgraph F of G is called a (g, f) -factor if one has $g(x) \leq d_F(x) \leq f(x)$ for each vertex x in G . An (f, f) -factor is said to be an f -factor. If G includes an r -factor for every $r : V(G) \rightarrow N$ which satisfies $g(x) \leq r(x) \leq f(x)$ for each vertex x in G and $r(V(G))$ is even, then we say that G admits all (g, f) -factors. Let $h : E(G) \rightarrow [0, 1]$ be a function. For any $x \in V(G)$, we denote the set of edges incident with x by $E(x)$. If $g(x) \leq \sum_{e \in E(x)} h(e) \leq f(x)$ holds for each vertex x in G , then we call graph F_h with vertex set $V(G)$ and edge set E_h a fractional (g, f) -factor of G with indicator function h , where $E_h = \{e : e \in E(G), h(e) > 0\}$. A fractional (f, f) -factor is called a fractional f -factor. If G contains a fractional r -factor for every $r : V(G) \rightarrow N$ with $g(x) \leq r(x) \leq f(x)$ for each vertex x in G , then we say that G admits all fractional (g, f) -factors. If $g(x) \equiv a$, $f(x) \equiv b$ and G admits

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all fractional (g, f) -factors, then we say that G contains all fractional $[a, b]$ -factors. Let H be a subgraph of G . If for every $r : V(G) \rightarrow N$ such that $g(x) \leq r(x) \leq f(x)$ for each vertex x in G , G includes a fractional r -factor F_h with $h(e) = 1$ for any $e \in E(H)$, then we say that G admits all fractional (g, f) -factors including H , where h is the indicator function of F_h . For any function $\varphi : V(G) \rightarrow N$, we define $\varphi(S) = \sum_{x \in S} \varphi(x)$ and $\varphi(\emptyset) = 0$. Especially, $d_G(S) = \sum_{x \in S} d_G(x)$.

Lu [3] first introduced the definition of all fractional (g, f) -factors, and obtained a necessary and sufficient condition for a graph to have all fractional (g, f) -factors, and posed a sufficient condition for the existence of all fractional $[a, b]$ -factors in graphs. Zhou and Sun [4] showed a neighborhood condition for a graph to have all fractional $[a, b]$ -factors, which is an extension of Lu's result [3]. Zhou, Bian and Sun [5] obtained a binding number condition for the existence of all fractional $[a, b]$ -factors in graphs. The following results on fractional (g, f) -factors and all all fractional (g, f) -factors are known.

Anstee [1] gave a necessary and sufficient condition for graphs to have fractional (g, f) -factors. Liu and Zhang [2] posed a new proof.

Theorem 1 (Anstee [1], Liu and Zhang [2]). Let G be a graph, and $g, f : V(G) \rightarrow Z^+$ be two functions with $g(x) \leq f(x)$ for each vertex x in G . Then G contains a fractional (g, f) -factor if and only if

$$f(S) + d_{G-S}(T) - g(T) \geq 0$$

for any subset S of $V(G)$, where $T = \{x : x \in V(G) - S, d_{G-S}(x) < g(x)\}$.

The following theorem is equivalent to Theorem 1.

Theorem 2. Let G be a graph, and $g, f : V(G) \rightarrow Z^+$ be two functions with $g(x) \leq f(x)$ for each vertex x in G . Then G contains a fractional (g, f) -factor if and only if

$$f(S) + d_{G-S}(T) - g(T) \geq 0$$

for all disjoint subsets S and T of $V(G)$.

Lu [3] showed a characterization of graphs having all fractional (g, f) -factors.

Theorem 3 (Lu [3]). Let G be a graph and $g, f : V(G) \rightarrow Z^+$ be two functions with $g(x) \leq f(x)$ for each vertex x in G . Then G admits all fractional (g, f) -factors if and only if

$$g(S) + d_{G-S}(T) - f(T) \geq 0$$

for any subset S of $V(G)$, where $T = \{x : x \in V(G) - S, d_{G-S}(x) < f(x)\}$.

In this paper, we study the existence of all fractional (g, f) -factors including any given subgraph in graphs, and pose some new results which are shown in the following.

Theorem 4. Let G be a graph and $g, f : V(G) \rightarrow Z^+$ be two functions such that $g(x) \leq f(x)$ for each vertex x in G . Let H be a subgraph of G . Then G has all fractional (g, f) -factors including H if and only if

$$g(S) + d_{G-S}(T) - f(T) \geq d_H(S) - e_H(S, T)$$

for all disjoint subset S and T of $V(G)$.

Theorem 5. Let G be a graph, H be a subgraph of G , and $g, f : V(G) \rightarrow Z^+$ be two functions with $d_H(x) \leq g(x) \leq f(x) \leq d_G(x)$ for each vertex x in G . If $(g(x) - d_H(x))d_G(y) \geq (d_G(x) - d_H(x))f(y)$ holds for any $x, y \in V(G)$, then G has all fractional (g, f) -factors including H .

If $E(H) = \emptyset$ in Theorem 5, then we obtain the following corollary.

Corollary 6. Let G be a graph, and $g, f : V(G) \rightarrow Z^+$ be two functions with $g(x) \leq f(x) \leq d_G(x)$ for each vertex x in G . If $g(x)d_G(y) \geq d_G(x)f(y)$ holds for any $x, y \in V(G)$, then G contains all fractional (g, f) -factors.

2 The proof of Theorem 4

Proof of Theorem 4. We first verify this sufficiency. Let $r : V(G) \rightarrow Z^+$ be an arbitrary integer-valued function such that $g(x) \leq r(x) \leq f(x)$ for each $x \in V(G)$. According to the definition of all fractional (g, f) -factors including H , we need only to verify that G admits a fractional r -factor including H , that is, we need only to verify that G admits a fractional r' -factor excluding H , where $r'(x) = d_G(x) - r(x)$. Let $G' = G - E(H)$. Thus, we need only to prove that G' admits a fractional r' -factor.

For any disjoint subsets S and T of $V(G)$,

$$g(S) + d_{G-S}(T) - f(T) \geq d_H(S) - e_H(S, T),$$

and so,

$$g(T) + d_{G-T}(S) - f(S) - d_H(T) + e_H(S, T) \geq 0. \quad (1)$$

It follows from (1) that

$$\begin{aligned} r'(S) + d_{G'-S}(T) - r'(T) &= r'(S) + d_{G-S}(T) - r'(T) - d_H(T) + e_H(S, T) \\ &= d_G(S) - r(S) + d_{G-S}(T) - d_G(T) + r(T) - d_H(T) + e_H(S, T) \\ &\geq d_G(S) - f(S) + d_{G-S}(T) - d_G(T) + g(T) - d_H(T) + e_H(S, T) \\ &= g(T) + d_{G-T}(S) - f(S) - d_H(T) + e_H(S, T) \geq 0. \end{aligned}$$

In terms of Theorem 2, G' admits a fractional r' -factor, that is, G has all fractional (g, f) -factors including H .

Now we verify the necessary. Conversely, we assume that there exist disjoint subsets S and T of $V(G)$ such that

$$g(S) + d_{G-S}(T) - f(T) < d_H(S) - e_H(S, T).$$

Let $r(x) = g(x)$ for any $x \in S$ and $r(y) = f(y)$ for any $y \in V(G) \setminus S$. Thus, we have

$$\begin{aligned} 0 &> g(S) + d_{G-S}(T) - f(T) - d_H(S) + e_H(S, T) \\ &= r(S) + d_{G-S}(T) - r(T) - d_H(S) + e_H(S, T). \end{aligned}$$

Set $r'(x) = d_G(x) - r(x)$ and $G' = G - E(H)$. Thus,

$$\begin{aligned} 0 &> r(S) + d_{G-S}(T) - r(T) - d_H(S) + e_H(S, T) \\ &= d_G(S) - r'(S) + d_{G'-S}(T) + d_H(T) - e_H(S, T) - d_G(T) + r'(T) - d_H(S) + e_H(S, T) \\ &= d'_G(S) + d_H(S) - r'(S) + d_{G'-S}(T) + d_H(T) - d'_G(T) - d_H(T) + r'(T) - d_H(S) \\ &= r'(T) + d_{G'-T}(S) - r'(S), \end{aligned}$$

which implies that G' has no fractional r' -factor. (Otherwise, $r'(A) + d_{G'-A}(B) - r'(B) \geq 0$ for all disjoint subsets A and B of $V(G)$ by Theorem 2. Set $A = T$ and $B = S$. Thus, we obtain $r'(T) + d_{G'-T}(S) - r'(S) \geq 0$, a contradiction.) And so, G has no fractional r' -factor excluding H , that is, G has no fractional r -factor including H . Hence, G has no all fractional (g, f) -factors excluding H , a contradiction. This finishes the proof of Theorem 4. \square

3 The proof of Theorem 5

Proof of Theorem 5. According to Theorem 4, we need only to verify that

$$g(S) + d_{G-S}(T) - f(T) \geq d_H(S) - e_H(S, T)$$

for all disjoint subsets S and T of $V(G)$.

If $T = \emptyset$, then we have

$$g(S) + d_{G-S}(T) - f(T) = g(S) \geq d_H(S) = d_H(S) - e_H(S, T).$$

In the following, we assume that $T \neq \emptyset$. Note that $(g(x) - d_H(x))d_G(y) \geq (d_G(x) - d_H(x))f(y)$ holds for any $x, y \in V(G)$, that is, $g(x)d_G(y) \geq d_G(x)f(y) + d_H(x)(d_G(y) - f(y))$ holds for any $x, y \in V(G)$. Hence, we have

$$\left(\sum_{x \in S} g(x)\right) \left(\sum_{y \in T} d_G(y)\right) \geq \left(\sum_{x \in S} d_G(x)\right) \left(\sum_{y \in T} f(y)\right) + \left(\sum_{x \in S} d_H(x)\right) \left(\sum_{y \in T} (d_G(y) - f(y))\right),$$

that is,

$$g(S)d_G(T) \geq d_G(S)f(T) + d_H(S)(d_G(T) - f(T)). \quad (2)$$

We write $U = V(G) \setminus (S \cup T)$. Then we obtain

$$\begin{aligned} d_G(S) &= e_G(S, T) + e_G(S, S) + e_G(S, U) \\ &\geq e_G(S, T) + e_H(S, S) + e_G(S, U) \\ &= e_G(S, T) + d_H(S) - e_H(S, T) - e_H(S, U) + e_G(S, U) \\ &\geq e_G(S, T) + d_H(S) - e_H(S, T) \\ &= d_G(T) - d_{G-S}(T) + d_H(S) - e_H(S, T), \end{aligned}$$

which implies

$$d_G(S) - d_G(T) \geq -d_{G-S}(T) + d_H(S) - e_H(S, T). \quad (3)$$

In terms of (2) and (3), we have

$$\begin{aligned} &d_G(T)(g(S) + d_{G-S}(T) - f(T) - d_H(S) + e_H(S, T)) \\ &= d_G(T)g(S) + d_G(T)d_{G-S}(T) - d_G(T)f(T) - d_G(T)d_H(S) + d_G(T)e_H(S, T) \\ &\geq d_G(S)f(T) + d_H(S)(d_G(T) - f(T)) + d_G(T)d_{G-S}(T) - d_G(T)f(T) \\ &\quad - d_G(T)d_H(S) + d_G(T)e_H(S, T) \\ &= f(T)(d_G(S) - d_G(T)) + d_G(T)d_{G-S}(T) - d_H(S)f(T) + d_G(T)e_H(S, T) \\ &\geq f(T)(-d_{G-S}(T) + d_H(S) - e_H(S, T)) + d_G(T)d_{G-S}(T) - d_H(S)f(T) + d_G(T)e_H(S, T) \\ &= (d_{G-S}(T) + e_H(S, T))(d_G(T) - f(T)) \geq 0. \end{aligned}$$

Combining this with $d_G(T) \geq f(T) \geq |T| \geq 1$, we obtain

$$g(S) + d_{G-S}(T) - f(T) \geq d_H(S) - e_H(S, T).$$

Theorem 5 is proved. \square

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